Homework 2

COMS W3261, Summer A 2022

This homework is due **Monday, 6/5/2023, at 11:59pm EST**. Submit to GradeScope (course code: K3VK75). If you use late days, the absolute latest we can accept a submission is Friday at 11:59 PM EST.

Grading policy reminder: LaTeX is preferred, but neatly typed or handwritten solutions are acceptable.[1] Feel free to use the .tex file for the homework as a template to write up your answers, or use the template posted on the course website. Your TAs may dock points for indecipherable writing.

Proofs should be complete; that is, include enough information that a reader can clearly tell that the argument is rigorous.

If a question is ambiguous, please state your assumptions. This way, we can give you credit for correct work. (Even better, post on Ed so that we can resolve the ambiguity.)

LaTeX resources.

- Detexify is a nice tool that lets you draw a symbol and returns the LaTeX codes for similar symbols.
- The tool Table Generator makes building tables in LaTeX much easier.
- The tool Finite State Machine Designer may be useful for drawing automata. See also this example (PDF) (.tex) of how to make fancy edges (courtesy of Eumin Hong).
- The website mathcha.io allows you to draw diagrams and convert them to LaTeX code.
- To use the previous drawing tools (and for most drawing in LaTeX), you'll need to use the package Tikz (add the command "\usepackage{tikz}" to the preamble of your .tex file to import the package).
- This tutorial is a helpful guide to positioning figures.

[1] The website Overleaf (essentially Google Docs for LaTeX) may make compiling and organizing your .tex files easier. Here's a quick tutorial.

Problem 1 (14 points)

Examine each of the following regular expressions and write down the language it describes using set notation or 1-2 sentences. (Example: $01^+ = \{w \mid w$ consists of a single 0 followed by at least one 1$\}$ or "This regular expression describes the language of strings that consist of a single 0 followed by at least one 1".)

1. (1 point.) $\Sigma\Sigma\Sigma\Sigma\Sigma\Sigma 1$, where $\Sigma = \{0, 1\}$.

   All 6-character binary strings ending in a 1.

2. (1 point.) $A^*(a \cup e \cup i \cup o \cup u)A^*$, where A denotes the alphabet $\{a, b, c, \ldots, x, y, z\}$, that is, the lowercase roman letters.

   All strings over A that contain at least one vowel (element of $\{a, e, i, o, u\}$).

3. (1 point) $(\Sigma 1)^* \cup (1\Sigma)^*$, where $\Sigma = \{0, 1\}$.

   All even length binary strings (including ϵ) in which every other character (starting with either the first or second character) is a 1.

4. (1 point) $(0^*10^*1)^*0^*$, where $\Sigma = \{0, 1\}$.

   All binary strings with an even number of 1's (including ϵ).

5. (1 point) $1(0 \cup 1 \cup 2 \cup 3 \cup 4)DDD \cup (00501 \cup 00544 \cup 06390)$, where D is the decimal alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

   Bonus (0 points): to what real-world set does this regular expression correspond?

   5-digit strings starting with 10, 11, 12, 13, or 14, and also the three strings 00501, 00544, and 06390. This set is the ZIP codes of New York State (almost exactly).

Write regular expressions that evaluate to the languages given.

6. (1 point). Odd, positive integers written in base 10. You may reference the set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ if you want or define your own alphabet(s).

   $D^*(1 \cup 3 \cup 5 \cup 7 \cup 9)$, or D^*O for $O = \{1, 3, 5, 7, 9\}$, or equivalents.

7. (1 point.) All strings over $\{0, 1\}$ that have an even number of 0's.

   $(1^*01^*0)^*1^*$, or $1^*(01^*01^*)^*$, or $1^* \cup (1^*01^*01^*)^*$, or equivalents.

8. (3 points.) All strings over $\{a, b\}$, including the empty string, that have no b's next to each other. (For example: ϵ, b, aaa, bab, $abaab$ are included, abb and $bbab$ are not.)

   Note that we lack a way to directly exclude certain (sub)strings: the set difference operator ($\setminus$ or $-$) isn't part of our definition of a regular expression.

   $b \cup (a^+b)^*a^* \cup a^*(ba^+)^*$, or equivalents.

   This expression divides strings with no b's next to each other into three classes: (1) strings of only b's, of which b is the only possibility, (2) strings in which every b is preceded by at least

one a, and (3) strings in which every b is followed by at least one a. The union of these three (not completely disjoint) classes gives the language.

Other regular expressions may work: to test yours, make sure it accepts ϵ, b, all strings of only a's, and the strings ab, ba, and aba; also make sure you can't generate two consecutive b's.

9. (4 points.) Consider the alphabet $\Sigma = \{\rightarrow, \underset{0}{\rightarrow}, \underset{1}{\rightarrow}, \bigcirc, \odot\}$. Here $\rightarrow$ indicates a start symbol, $\underset{0}{\rightarrow}$ and $\underset{1}{\rightarrow}$ indicate labeled transitions, $\bigcirc$ indicates a reject state, and $\odot$ indicates an accept state. (So the string $\rightarrow \bigcirc \underset{0}{\rightarrow} \bigcirc \underset{0}{\rightarrow} \odot$ corresponds to an NFA that accepts only the string '00', and $\rightarrow \odot$ corresponds to an NFA that accepts only the string ϵ.)

   Write

   - a regular expression that generates each NFA that accepts a single string of length 2. ($\rightarrow \bigcirc \underset{0}{\rightarrow} \bigcirc \underset{0}{\rightarrow} \odot$ is such an NFA, so it should be generated by your regular expression.)
     $\rightarrow \bigcirc(\underset{0}{\rightarrow} \cup \underset{1}{\rightarrow}) \bigcirc (\underset{0}{\rightarrow} \cup \underset{1}{\rightarrow})\odot$.
   - a regular expression that generates each NFA that accepts a single string of only zeroes. Don't count ϵ as a string of only zeroes. ($\rightarrow \bigcirc \underset{0}{\rightarrow} \bigcirc \underset{0}{\rightarrow} \odot$ is such an NFA, so it should be generated by your regular expression.)
     $\rightarrow (\bigcirc \underset{0}{\rightarrow})^{+}\odot$.

Rationale: The goal of this question is to make sure you're comfortable interpreting and building regular expressions (and reading NFAs).

References: Sipser pp. 63-66 (regular expressions, Lightning Review 3 (Regular Expressions); for 1.10 Sipser pp. 47-52; Lightning Review 2 (NFAs).

Problem 2 (8 points)

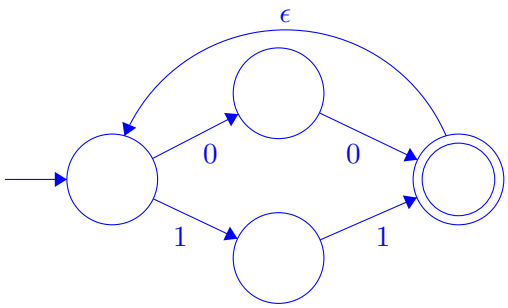
1. (3 points). Using the binary alphabet $\Sigma = \{0, 1\}$, draw a state diagram for an NFA **with at most 4 states** that recognizes the regular expression

$$(00 \cup 11)(00 \cup 11)^*.$$

Explain in words why your NFA recognizes the language specified.

[Hint: Feel free to use the techniques we used in our closure proofs, for example, concatenating NFAs. These techniques don't necessarily produce an NFA with the minimum number of states, so you may need to simplify afterwards. Alternatively, start by understanding and/or simplifying the regular expression, then directly build an NFA that recognizes the same language. Don't forget to check strings!]

First, we observe that $(00 \cup 11)(00 \cup 11)^* = (00 \cup 11)^+$, that is, all strings made up of the building blocks '00' and '11' (except the empty string). We can build the NFA directly.



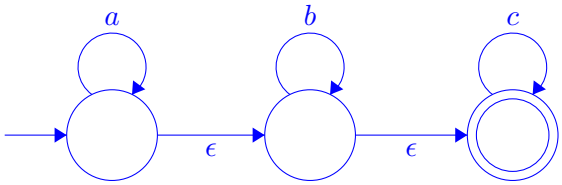
2. (5 points). Using the alphabet $\{a, b, c\}$, draw a state diagram for an NFA **with at most 3 states** that recognizes the regular expression

$$\epsilon \cup a^+c^+ \cup a^*b^*c^*.$$

Explain in words why your NFA recognizes the language specified.

The key here is to recognize that both $\{\epsilon\}$ and a^+c^+ are subsets of $a^*b^*c^*$: that is, ϵ and any string generated by a^+c^+ are also generated by $a^*b^*c^*$.

Simplifying the regular expression to $a^*b^*c^*$, we can write the following NFA:



Some modifications, like making all three states accept states, may also work.

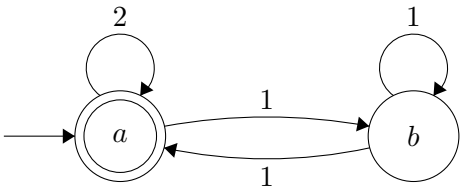
Rationale: The goal of this question is to practice building NFAs that recognize languages defined with regular operations, and simplifying the operations of NFAs.

References: Sipser pp. 63-66 (NFAs), Lightning Review 2 (NFAs); Sipser pp. 59-63 (combining NFAs to recognize languages built with regular operations).

Problem 3 (9 points)

1. (6 points.) In class, we described a systematic procedure for converting an NFA to a DFA by creating states corresponding to the *power set* of the original NFA state set.

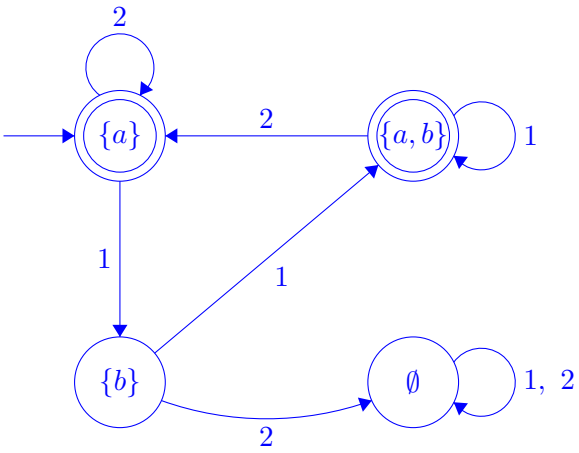
   The NFA below has two states, a and b, and uses the alphabet $1, 2$. Draw the state diagram of an equivalent DFA that uses the states $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\}$.



2. (3 points.) Describe the language recognized by the pictured NFA (and the DFA you've just drawn) and justify your reasoning using either automaton.

The equivalent DFA is drawn below. We observe (either by looking at the NFA, or at the equivalent DFA, generated procedurally) that this automaton accepts every string unless it contains an 'isolated 1': that is, a 1 character not adjacent to any other 1 character.

To see this, observe by looking at the DFA that we reject if we read in a 1 at the beginning of the string or after reading in a 2 (that is, when we are in state $\{a\}$), and then either the string ends or we read in another 2.



Rationale: The goal of this question is to practice thinking about DFAs that simulate other automata, as well as the specific process of converting an NFA into an equivalent DFA.

References: See Sipser pp. 54-58 (Converting DFAs to NFAs), Example 2 (Converting NFAs to DFAs).

Problem 4* (1 bonus point)

On page 87, Exercise 1.24, Sipser defines the *Finite State Transducer*, a type of DFA whose output is a string.

- (0.5 points). Read Exercise 1.24 and compute the output of the FST T_2 on the string 'baaabb'.

  The output is 101110.

- (0.5 points). Read Exercise 1.25 and write down a formal definition of an FST.

  An FST can be written as the 5-tuple $(Q, \Sigma, \Gamma, q_0, \delta)$. Here, Q denotes a finite set of states, Σ denotes the *input alphabet* (that is, the set of valid symbols that the machine can read in as input), Γ denotes the *output alphabet* (that is, the set of valid symbols that the machine can write as output), q_0 denotes the start state, and $\delta : Q \times \Sigma \to Q \times \Gamma$ is a transition function that takes a (state, input character) pair as input and returns a (state, output character) pair as output. Note that we don't need to specify accept states, as the FST returns a string instead of accepting or rejecting.

Rationale: Optional, just for fun. Bonus points will be added to your score on this HW, which is out of 31; they don't count for more or less than regular points.