COMS 3261 Homework 0

Summer A 2022

This problem set is OPTIONAL. The purpose is to test yourself and review discrete mathematics concepts that will be essential for the course.

Grading policy reminder: LaTeX is preferred, but neatly typed or handwritten solutions are acceptable.[1] I recommend using the .tex file for the homework as a template to write up your answers. Your TAs may dock points for indecipherable writing.

Proofs should be complete; that is, include enough information that a reader can clearly tell that the argument is rigorous.

If a question is ambiguous, please state your assumptions. This way, we can give you credit for correct work. (Even better, post on Ed so that we can resolve the ambiguity.)

The tool `http://madebyevan.com/fsm/` may be useful for drawing finite state machines. Detexify is a nice tool that lets you draw a symbol and returns the LaTeX codes for similar symbols. The website mathcha.io allows you to draw diagrams and convert them to LaTeX code. (You'll need to add the command "\usepackage{tikz}" to the preamble of your .tex file to import the package tikz.)

[1] The website Overleaf (essentially Google Docs for LaTeX) may make compiling and organizing your .tex files easier. Here's a quick tutorial.

1 Problem 1 (0 points).

1. Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

   (a) $\{1, 3, 5, 7...\}$.
   This is the set of positive odd integers (equivalently, odd natural numbers.)

   (b) $\{n \mid n = 2m \text{ for some } m \in \mathbb{N}\}$. (Here $\mathbb{N}$ denotes the natural numbers $\{1, 2, \dots\}$.)
   This is the set of positive even integers (equivalently, even natural numbers).

   (c) $\{w \mid w \text{ is a } \textit{binary string} \text{ (a string of 0's and 1's) and } w \text{ equals the reverse of } w\}$.
   This is the set of palindromes over the binary alphabet $\{0, 1\}$, including elements like '0110', '1010101', '1111', '0', and the empty string $\epsilon =$ ''.

2. Write formal descriptions of the following sets.

   (a) The set containing the numbers 1, 10, and 100.
   $\{1, 10, 100\}$.

   (b) The set containing all natural numbers less than 5.
   $\{1, 2, 3, 4\}$ or $\{n \mid n \in \mathbb{N}, n < 5\}$.

   (c) The set containing nothing at all. (There are two ways to write this: the "natural" way and using a certain special symbol.)
   $\{\}$ or $\emptyset$.

3. If the set A has $|A|$ elements and the set B has $|B|$ elements, how many elements are in $A \times B$? (Here $\times$ denotes the *Cartesian product*: $A \times B$ represents the set of all ordered pairs (a, b) consisting of one element $a \in A$ and one element $b \in B$.)

   $|A| \cdot |B|$. This is because every possible choice of one element from A and one element from B corresponds to a distinct member of $A \times B$. Note that if $A = B = \{1, 2\}$, $A \times B = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$; that is, $|A \times B| = 4$ because the tuples $(1, 2)$ and $(2, 1)$ are distinct.

4. If C is a set with $|C|$ elements, how many elements are in the power set $\mathcal{P}(C)$? (The power set $\mathcal{P}(C)$ is the set containing all subsets of C.)

   $2^{|C|}$. To see this, observe that if we want to construct a subset of C, we can make $|C|$ independent binary choices about whether to include each element $c \in C$. Note that both $\emptyset$ and C itself are subsets of C.

5. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. What is $A \cap B$? $A \cup B$? $A \setminus B$ (also written $A - B$)?[2] What is the complement $\overline{A}$ of the set A, considered with respect to the universe of integers $\mathbb{Z}$?

   $A \cap B = \{1, 3, 5\}$. $A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$. $A \setminus B = \{2, 4\}$. There are a few ways to write $\overline{A}$: among them, we could denote this set by $\mathbb{Z} \setminus A$, or by $\{n \in \mathbb{Z} \mid n < 1 \text{ or } n > 5\}$.

[2] You may have seen $A \setminus B$ defined only when B is a subset of A ($B \subseteq A$.) In this course, we'll use $A \setminus B$ in a slightly broader sense to mean "all the elements of A, minus any elements in A that are also in B."

6. We use parentheses () to distinguish *sequences*, which care about the order of their elements, from sets. Of course, we can also make sets of sequences. What is $\{(a, b), \{a, b\}\} \cup \{(b, a), \{b, a\}\}$?

   This is a little tricky. Here we have two sets, each of which *contains* one ordered pair (tuple) and one set as an element. (Yes, we can have sets that contain different types of objects, although the results aren't always pleasant.) (a, b) is different from (b, a), because we care about the order of tuples. Moreover, (a, b) is different from $\{a, b\}$, as the first object is a tuple and the second object is a set. (If this is confusing, imagine tuples and sets as different data structures. They both contain pointers to a and b, but the tuple stores additional information, namely an ordering.) However, $\{a, b\} = \{b, a\}$. Eliminating the duplicate element, the answer is $\{(a, b), (b, a), \{a, b\}\}$.

7. Prove that
$$A \cap ((B \cup \overline{A}) \cap \overline{B}) = \emptyset.$$

   There are a few ways to approach this problem. For me, it's intuitive to remember that the intersection operation (and the union operation) are commutative and associative, so this expression is asking us to evaluate the intersection of three sets: A, $\overline{B}$, and $(B \cup \overline{A})$.

   Any element in both A and $\overline{B}$ clearly can't be in B or $\overline{A}$, and thus is not contained in $(B \cup \overline{A})$. Thus no element is contained in the intersection of all three sets A, $\overline{B}$, and $(B \cup \overline{A})$.

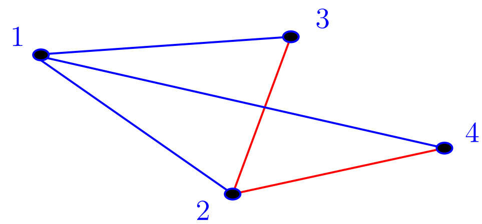
   A second way to think about this is to evaluate the terms inside the parentheses first: any element in $\overline{B}$ that's also in $B \cup \overline{A}$ must be in $\overline{A}$. $A \cap \overline{A} = \emptyset$.

Rationale: The goal of this question is to make sure you're comfortable with the idea of sets, set notation and set operations.

References: Sipser 0.2 pp. 3-6; Simple English Wiki: Set.

2 Problem 2 (0 points).

1. Consider the undirected graph $G = (V, E)$, where V, the set of nodes, is $\{1, 2, 3, 4\}$ and E, the set of edges, is $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\}$. Draw G. What are the degrees of each node? Indicate a path from node 3 to node 4 on the graph.

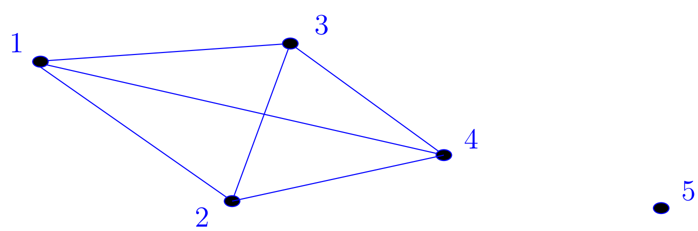
   

   Above is a picture of the graph (drawn in the package Tikz, using the helper website mathcha.io). Nodes 1 and 2 have degree 3, while nodes 3 and 4 have degree 2. One possible path between nodes 3 and 4 is indicated in red.

2. How many components (connected pieces) does the graph

   $$G = (\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}, \{2, 4\}\})$$

   contain? How many triangles does it contain?

   A picture of the graph is displayed above. There are two connected components (the subgraphs induced on $\{1, 2, 3, 4\}$ and $\{5\}$). There are 4 triangles (all 3-item subsets of $\{1, 2, 3, 4\}$).

3. Prove that every *tree* (connected graph without a cycle) that has $n \geq 1$ vertices has exactly $n - 1$ edges.

   We can prove this fact in a few different ways. For me, one intuitive way to do so is by visualizing a breadth-first search. Run BFS starting at an arbitrary node, which we'll call v_1. By assumption, our BFS reaches every node (otherwise, our graph is not connected) and never discovers the same node twice via different edges (otherwise, our graph contains a cycle.) As a result, our BFS must discover exactly $n - 1$ nodes in addition to v_1 by exploring exactly $n - 1$ fresh edges.

Rationale: The goal of this question is to make sure you're comfortable with graphs and proofs that refer to generic graphs.

References: Sipser 0.2 pp. 10-13; Wiki: Graph.

3 Problem 3 (0 points).

1. Given $f : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$ where f is defined as:

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is even} \\ x-1 & \text{if } x \text{ is odd} \end{cases}$$

Prove or provide a counterexample that f is

(a) one-to-one (injective),

Clearly, f maps even numbers to odd numbers and odd numbers to even numbers, so $f(x) \neq f(y)$ if x is even and y is odd. If x_1, x_2 are even, then $f(x_1) = f(x_2)$ implies $x_1 + 1 = x_2 + 1$, so $x_1 = x_2$. The argument is similar for y_1, y_2 odd. Thus f is one-to-one.

(b) onto (surjective),

If $x \in \mathbb{N} \cup \{0\}$ is even, then $f(x+1) = x$ (and $x + 1 \in \mathbb{N} \cup \{0\}$). Likewise, if $x \in \mathbb{N} \cup \{0\}$ is odd, then $f(x-1) = x$. $x - 1 \in \mathbb{N} \cup \{0\}$ as the smallest odd number is 1 and $1 - 1 = 0$ is in the domain. Thus f is onto.

(c) and bijective (both one-to-one and onto).

The first two parts immediately imply that f is bijective.

Rationale: The goal of this question is to make sure you're comfortable thinking about discrete functions.
References: Sipser 0.2 pp. 7-10, Wiki: One-to-one and Onto, Wiki: Relations.

4 Problem 4 (0 points).

Prove using *contradiction* that for all integers n, if 5 divides n^2 then 5 divides n [Hint: what does it mean to not be divisible by 5?].

We want to show that if 5 divides n^2, 5 divides n. So we'll assume the opposite and try to contradict it. Specifically, let's suppose for contradiction that there exists an integer n such that 5 divides n^2 but 5 does not divide n.

By the fundamental theorem of arithmetic, we can decompose n into a unique product of prime powers

$$p_1^{k_1} p_2^{k_2} \dots p_\ell^{k_\ell},$$

and $n^2 = (p_1^{k_1} p_2^{k_2} \dots p_\ell^{k_\ell})^2$. If 5 divides n^2, then $5 = p_i$ for some $i \in \{1, 2, \dots, \ell\}$. However, this implies that 5 divides n, contradicting our assumption.

The hypothesis that there exists some integer n such that 5 divides n^2 but not n thus leads to a contradiction. Thus if 5 divides n^2, 5 divides n.

5 Problem 5 (0 points).

Prove that for any positive integer n, there exists a sequence of n consecutive positive composite integers. [Hint: try to construct such a sequence!]

We'd like to find a sequence of consecutive integers $x_1, x_2, x_3, \dots, x_n$ such that each element of the sequence is clearly divisible by some smaller number. Moreover, prime numbers are relatively abundant among small integers. If we want to find a sequence of composite integers that meets our conditions, we're going to have to consider very large numbers.

The key is to imagine a number that's the product of a bunch of consecutive numbers. For example, fix n and consider the product

$$k_n = \prod_{i=1}^{n} (i+1).$$

Because 2 divides k_n, 2 also divides $k_n + 2$. Because 3 divides k_n, 3 also divides $k_n + 3$. Continue this logic up to the conclusion that $n+1$ divides $k_n + n + 1$. Thus the sequence $(k_n + 2, k_n + 3, \dots, k_n + n + 1)$ is composed entirely of composite numbers.

Rationale: The goal of this question is to make sure you're comfortable with proof by contradiction and construction.

References: Sipser 0.4, pp. 21-22. See also Sipser 0.3 on strategies for finding proofs, and this Medium post that introduces proof by construction, contradiction and induction with examples.

6 Problem 6 (Proof by Induction).

1. Let $S(n) = 1 + 2 + \cdots + n$ be the sum of the first n natural numbers and let $C(n) = 1^3 + 2^3 + \cdots + n^3$ be the sum of the first n cubes. Prove the following equalities by induction on n to arrive at the curious conclusion that $C(n) = S(n)^2$ for every n.

   (a) $S(n) = \frac{1}{2}n(n+1)$.

   Base case: $S(1) = 1 = (1)(1+1)/2$.

   Inductive step: We'll prove that $S(k+1) = (k+1)(k+2)/2$ under the assumption that $S(k) = k(k+1)/2$. To see this, observe that

   $$\begin{aligned} S(k+1) &= 1 + 2 + \cdots + (k+1) && (1)\\ &= S(k) + (k+1) && (2)\\ &= k(k+1)/2 + (k+1) && (3)\\ &= (k+1)(k/2+1) = (k+1)(k+2)/2, && (4) \end{aligned}$$

   where we use our inductive hypothesis in the second line. The result follows by induction. (If you're relatively unfamiliar with proof by induction, observe that we can apply the argument from our inductive step to the base case, $k = 1$, to conclude that the statement holds for $S(2)$. With the $k = 2$ case in hand, we can conclude the statement holds for $S(3)$. And so on, for all natural numbers n.)

   (b) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n+1)^2$.

   Base case: $C(1) = 1 = (1+2+1)/4$.

   Inductive step: we'll show that the statement holds for $C(k+1)$ under the assumption that it holds for $C(k)$. We have

   $$\begin{aligned} C(k+1) &= C(k) + (k+1)^3 && (5)\\ &= (k^2)(k+1)^2/4 + (k+1)(k+1)^2 && (6)\\ &= (k^2+4k+4)(k+1)^2/4 && (7)\\ &= (k+2)^2(k+1)^2/4. && (8) \end{aligned}$$

2. Assume n is a positive integer. Use induction to prove the following for all natural numbers n:

   $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n\cdot(n+1)} = 1 - \frac{1}{n+1}.$$

   Base case: When $n = 1$, $1/(1\cdot 2) = 1 - 1/(1+1) = 1/2$.

   Inductive step: Assuming the statement holds for some natural number k, we'll prove it for $k+1$. We have

   $$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)\cdot(k+2)} = (1 - \frac{1}{k+1}) + \frac{1}{(k+1)\cdot(k+2)} \qquad (9)$$

$$= 1 - \frac{k+2}{(k+1)\cdot(k+2)} + \frac{1}{(k+1)\cdot(k+2)} \tag{10}$$

$$= 1 - \frac{k+1}{(k+1)\cdot(k+2)} = 1 - \frac{1}{k+2}. \tag{11}$$

Rationale: The goal of this question is to make sure you're comfortable with proof by induction.

References: Sipser 0.4, pp. 22-25. See also Sipser 0.3 on strategies for finding proofs, and this Medium post that introduces proof by construction, contradiction and induction with examples.