The Context-Free Pumping Lemma

Sipser 128–129.

$$C = \{a^i b^j c^k \mid 0 \le i \le j \le k\}$$

1. Assume that C is a CFL $\Rightarrow$ contradiction.

By our assumption, C satisfies the CFPL, which means the following is true:

"There exists some number p such that any suff. long string $s \in C$ with $|s| \ge p$ can be divided into $s = uvxyz$ satisfying
(1) for each $i \ge 0$, $uv^ixy^iz \in C$
(2) $|vy| > 0$
(3) $|vxy| \le p$."

2. To contradict: find $s \in C$ with $|s| \ge p$ that can't be divided in any way satisfying (1)-(3).

candidate 1: $s = abbc^p$. $\in C$, $|s| \ge p$.

$$\underbrace{a}_{u}\underbrace{b}_{v}\underbrace{bccc \ldots cc}_{x}\underbrace{c}_{y} \quad z = \varepsilon.$$

can be pumped! ☹

candidate 2: $s = a^p b^p c^p$

Suppose we divide $a^p b^p c^p = uvxyz$.

Case (1): v and y both contain only one type of alphabet symbol.

1a) no a's in v or y.

then: $uv^0xy^0z = uxz \notin C$, as this decreases the number of b's or c's.

1b) no b's in v or y.

If vy contains a's, then $uvvxyyz$ has more a's than b's.

If vy contains c's then uxz has fewer c's than b's.

1c) no c's in v or y

here $uvvxyyz$ contains more a's or b's than c's.

case (2): either v or y contains two kinds of alphabet symbol.

$$\underbrace{aa\cdots}_{u}\,\underbrace{aab}_{v}\underbrace{b\cdots}_{x}\underbrace{bb}_{y}\underbrace{cc\cdots cc}_{z}$$

$uvvxyyz$ now contains some symbols out of order.

So: any way of splitting my string $S = a^p b^p c^p$ into $uvxyz$ breaks one of my three conditions.

$\therefore$ C fails the CFPL

$\therefore$ C is not context-free. $\square$