Sipser pp. 216-220.

Known: $A_{TM} = \{ \langle M, w \rangle \mid M$ is a TM that accepts $w \}$
$E_{TM} = \{ \langle M \rangle \mid M$ is a TM that doesn't accept any string. $\}$ / undecidable.

Prop. $EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2$ are TMs and $L(M_1) = L(M_2) \}$.
is undecidable.

Proof. We'll assume that some decider T decides EQ_{TM}.
Given T, we'll build a machine S that decides E_{TM} as follows:

S: "On input $\langle M \rangle$.
1. S writes down $\langle M_{NO} \rangle$, M_{NO} is a simple TM that rejects all strings.
2. S runs $T(\langle M, M_{NO} \rangle)$.
If $T(\langle M, M_{NO} \rangle)$ accepts, then accept.
Otherwise, reject."

Contradiction, because E_{TM} is not decidable. Thus our assumption is false — EQ_{TM} is undecidable.

Prop. LOOP $= \{ \langle M, w \rangle \mid M$ is a TM that runs forever on $w \}$.

Proof. We assume there exists a recognizer R for LOOP and use it to build a decider for A_{TM}.
Given R,

S: "On input $\langle M, w \rangle$,
– simulate $M(w)$ and $R(\langle M, w \rangle)$ in parallel, alternating steps until one machine accepts.
– If $M(w)$ accepts, accept. If $R(\langle M, w \rangle)$ accepts, reject."
If $M(w)$ rejects, reject.

S decides A_{TM} (and halts on all input.). A_{TM} is not decidable so this is a contradiction. ■