Proving A_{TM} is Undecidable (via contradiction/paradox)

Sipser pp. 207 - 208.

Theorem: $A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM that accepts } w\}$ is undecidable.

Strategy: Assume A_{TM} is decidable and use decider as a subroutine in a "paradox machine" (find a contradiction.)

Proof. Assume A_{TM} is decidable, and let H decides A_{TM}.

So: $H(\langle M, w\rangle)$ accepts if and only if $M(w)$ accepts.

(*1) $H(\langle M, \langle M\rangle\rangle)$ accepts if and only if $M(\langle M\rangle)$ accepts.

Define a new decider P that works as follows:

P = "On input $\langle M\rangle$, where M is a TM,
(*2) run $H(\langle M, \langle M\rangle\rangle)$ and output the opposite of H."

Now - consider running $P(\langle P\rangle)$.

P will simulate $H(\langle P, \langle P\rangle\rangle)$.

If $\underbrace{H(\langle P, \langle P\rangle\rangle)}_{= P(\langle P\rangle) \text{ accepts}}$ accepts, (*1) $P(\langle P\rangle)$ rejects. (*2)

If $\underbrace{H(\langle P, \langle P\rangle\rangle)}_{= P(\langle P\rangle) \text{ rejects.}}$ rejects, $P(\langle P\rangle)$ accepts. ???

Contradiction! Thus A_{TM} is not decidable.