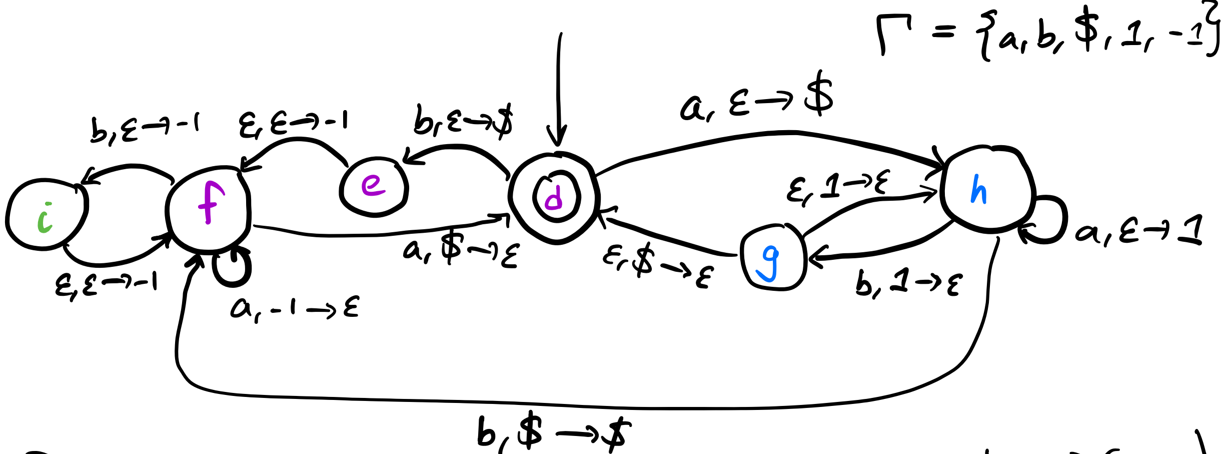
$\Sigma = \{a, b\}$

$\Gamma = \{a, b, \$, 1, -1\}$



Puzzle: does this PDA accept:

(a, b ⟶ c
read, pop ⟶ push)

ε? ✓

aab? ✓

baa? ✓

ab? ✗

aba? ✓

aabbaa? ✓

bbaaa? ✗ $\boxed{-1 / \$}$

bbaaaa? ✓

What language does this PDA recognize?

pop -1

$\{w \in \{a,b\}^* \mid w$ has twice as many a's as b's.$\}$

On baa: d, $\boxed{\$}$, e, $\boxed{-1 / \$}$, f, $\boxed{\$}$, f, ___, d ✓

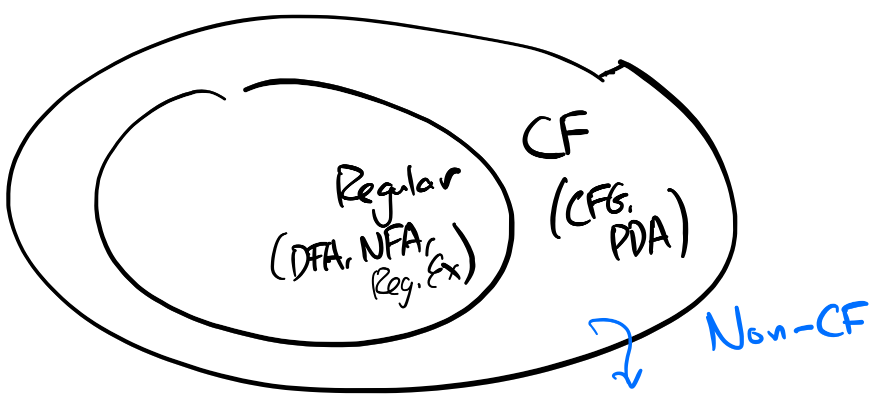
b, ε → $ a, -1 → ε   a, $ → ε

On aba: $\boxed{\$}$, $\boxed{\$}$, ___ ✓

aab: $\boxed{\$}$ $\boxed{1 / \$}$ $\boxed{\$}$ ___ ✓

$\delta : Q \times \Gamma \longrightarrow Q \times \Gamma$

state, stack symbol, new state, push



Today: 1. A pumping lemma for CFLs.

2. TURING MACHINES (!)

Coming up.

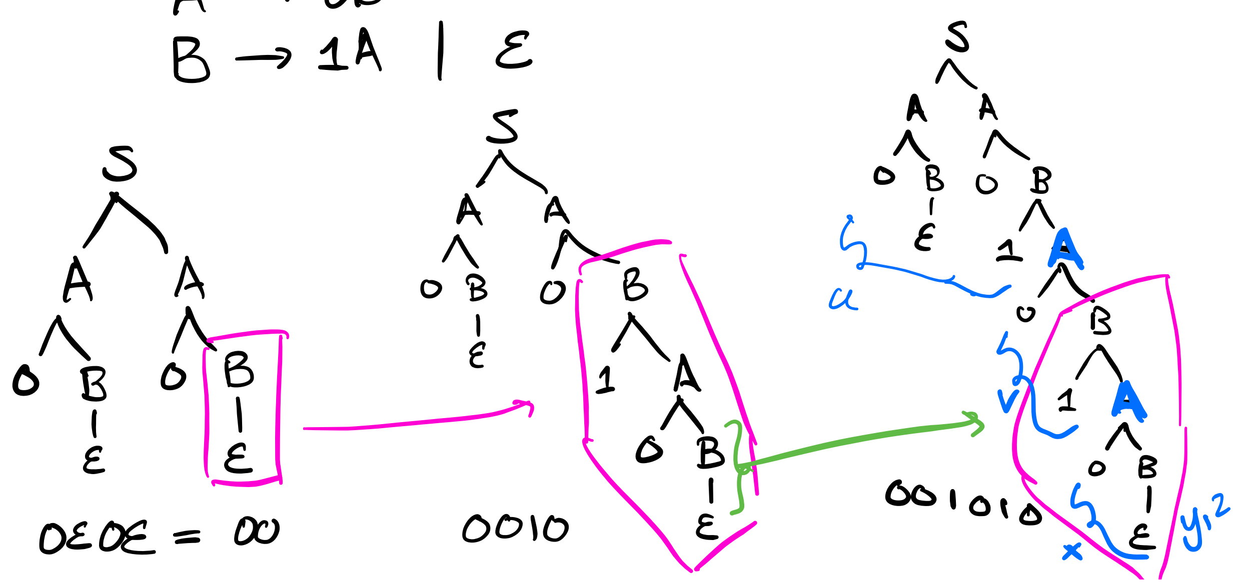
- This week: Turing Machines!
  (M,W)
- Next week: Undecidable, unrecognizable languages
  (W,F) halting problem
  simulations
  make-up will be recorded
- Last week: Time complexity, space complexity,
  (M,W) P, NP, NP-completeness,
  not on final info. theory.

(R-F: open book/note take home final)

Short survey on HW section of website - do for Weds (5 mins)

1. Context-Free Pumping Lemma

$$S \to AA$$
$$A \to 0B$$
$$B \to 1A \mid \varepsilon$$



- Can replace the subtree descending from B w/ any other subtree descending from B to get a new, valid parse tree.
- If the language of our grammar is infinite, the derivations of very long strings must contain repeated variables.

Theorem (CFPL). For any context-free language L, there exists some number p, such that for any $w \in L$ w/ $|w| \geq p$, w can be divided into five substrings $w = uvxyz$ such that

(1) $uv^ixy^iz \in L$, for $i \geq 0$. $\left(uxz, uvxyz, uvvxyyz, uvvvxyyyz, \ldots \right)$

(2) $|vy| > 0$

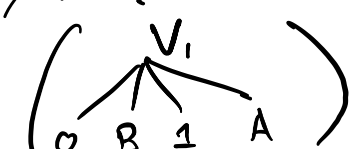
(3) $|vxy| \leq p$.

Proof. Let L be a CFL.

If L is finite: set $p >$ length of longest string in L; trivially true.

If L is infinite:

Let $G = (V, \Sigma, R, S)$ be a CFG for L, and let b denote the 'branching factor' of G: largest # of symbols on the righthand side of any rule.



(*) We can derive at most b^d terminals with a parse tree of depth $\leq d$. (d = length of longest path from root to leaf.)

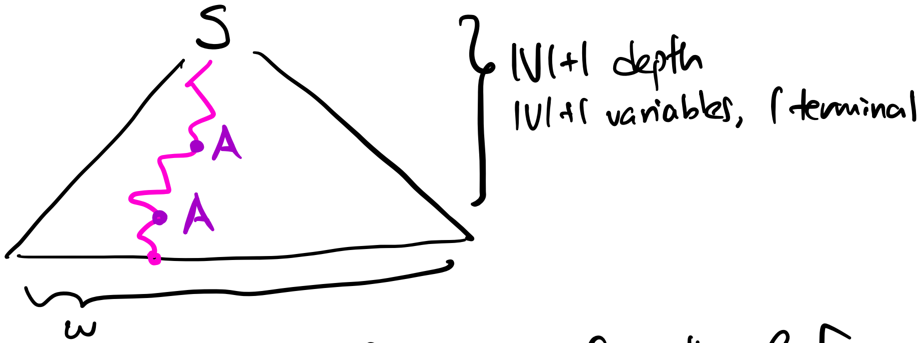
$d=2$

$b=4$

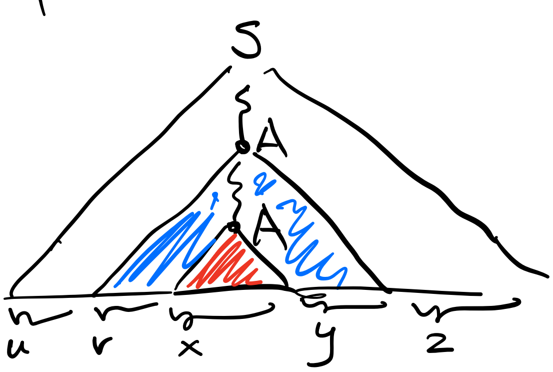
leaves $< b^d = 4^2$.

Let $p = b^{|V|+1}$. Fix a minimum-height parse tree deriving a string w of length $w \geq p = b^{|V|+1}$.

By (*), this parse tree has depth $\geq |V|+1$. So: there exists a path P from root to leaf of length $|V|+1$, with at least $|V|+1$ variables.



So: if we traverse P starting from the leaf, we find a repeated variable A within $|V|+1$ steps.

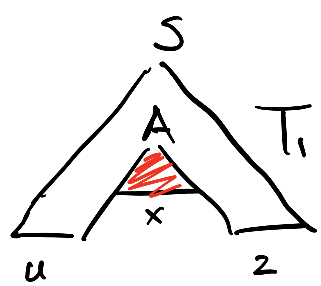


- Let x be the substring of w derived from the lower A.
- Let v, y be the other substrings derived from the higher A.
- Let u, z be the remaining substrings

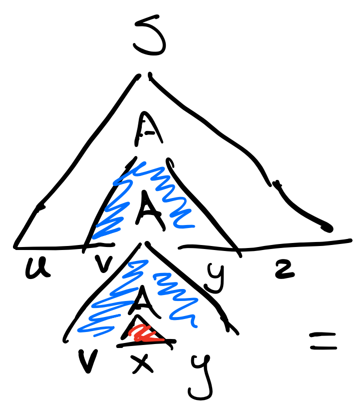
Claims:

(1) $uv^ixy^iz \in L$ for any $i \geq 0$.

Proof:
by copy paste.



$= uxz$



$= uvvxyyz$

(2) $|vy| > 0$.

Proof: we assumed our tree was minimal — but if $v, y = \varepsilon$, then the tree generating uxz would generate $uvxyz$

while being smaller.

(3) $|vxy| \le p$.

Proof: height of the higher $A \le |V|+1$ edges. Thus the number of terminals derived is $|vxy| \le b^{|V|+1} = p$. ∎

(Sipser p.125-127).

Proof: $B = \{a^n b^n c^n \mid n \ge 0\}$ is not context-free.

Assume for contradiction that B is CF. Then, by CFPL, there exists a number p such that any string $s \in B$, $|s| \ge p$ can be pumped; that is, can be subdivided into substrings $s = uvxyz$ such that:

(1) $uv^ixy^iz \in B$ for all $i \ge 0$,
(2) $|vy| > 0$,
(3) $|vxy| \le p$.

$uvxyz = a^p b^p c^p$
$uvvxyyz \Rightarrow$ more chars

We'll choose a contradiction string $s = a^p b^p c^p$, and we'll show that no way of dividing it satisfies (1), (2), and (3).

Case 1: vxy contains ~~only a's~~. only one type of character. In this case, uv^2xy^2z contains too many of this one character, if $|vy| > 0$. (example: v, y contain a's, $uv^2xy^2z = a^{p+|vy|} b^p c^p$).

Case 2: vxy contains two types of character. By (3), only a's and b's, or b's and c's, is possible.

Assume vxy contains a's and b's w.l.o.g., in which case uv^2xy^2z has too few c's. ∎

Thus s can't be pumped, which is a contradiction ⇒ B is non-CF.

↳ Or: uv^2xy^2z has at least one more a or one more b ⟹ too few c's.

— Back at 2:57 —

$$D = \{ww \mid w \in \{0,1\}^*\}.$$

$0^p1^p0^{2p}1^p0^p$

Given contradiction string $s = 0^p1^p0^p1^p$, what cases can help us argue that no partition $s = uvxyz$ satisfies
(1) $uv^ixy^iz \in D$ for $i \geq 0$, (2) $|vy| > 0$, (3) $|vxy| \leq p$.

(Bonus – why is 0^p10^p1 not a good contradiction string?)

Case 1: either v or y contains both 1's and 0's.
↳ If $|vxy| \leq p$, then $vxy \leq 0^p1^p0^p1^p$

Say $v = 0^i1^j$, and $vxy \leq$ the first 0^p1^p chunk.
Now, $uv^2xy^2z = 0^{p-i}\underbrace{0^i1^j}_{v}\underbrace{0^i1^j}_{v}1^{p-j+|y|}0^p1^p$

$0^p1^j0^i1^{p+|y|}0^p1^p \notin D.$

⟹ plausible case analysis, potentially long.

Case 1': vxy all one character

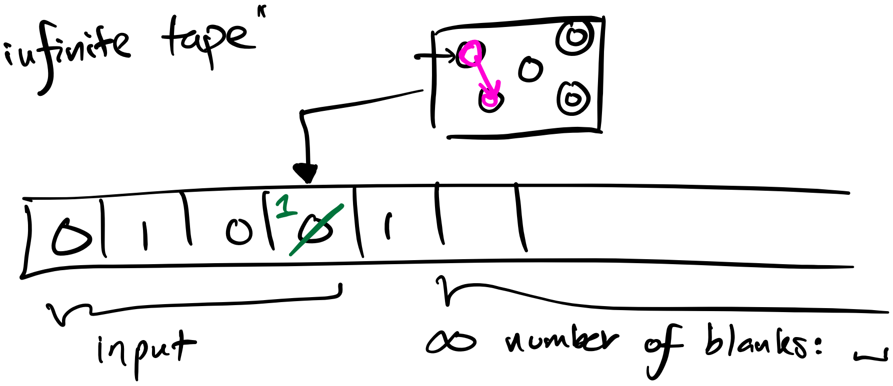
Case 2: vxy contains 0's and 1's
↳ 2.1: v, y each contain only one type of character (e.g. v all 0's, y all 1's).
↳ 2.2:

Moral of our story — case analysis nontrivial here — pick your cases with care.

2. Turing Machines

<u>Idea:</u> Automaton with Random-Access Memory.

the "infinite tape"



Every step of computation:

1) read an input of the tape
2) transition to a new (internal) state
3) rewriting current tape square
4) move one square L or R.

<u>Def.</u> A TM is a 7-tuple

$$(Q, \Sigma, \Gamma, \delta, q_0, \underbrace{q_{accept}, q_{reject}}_{\text{special accept/reject states.}}).$$

- Q: a finite set of states
- Σ: input alphabet
- Γ: tape alphabet (incl. Σ, ␣),
- q_0: start state
- q_{accept}, q_{reject}: special accept/reject states.

TM halts immediately on reaching q_{acc}, q_{rej}.

transition function: $\delta : Q \times \Gamma \longrightarrow \underbrace{Q}_{\text{new state,}} \times \underbrace{\Gamma}_{\text{what to write}} \times \underbrace{\{L, R\}}_{\text{where to move (on the tape)}}$

$\Sigma \cup \{␣\} \subseteq \Gamma$

$\cancel{1}$ i

Def (TM computation).

Start: input $\omega = \omega_1 \omega_2 \cdots \omega_n$, $\omega_i \in \Sigma$, on the left side of the tape. Tape head points at ω_1.

As a configuration, we can write this

$$\underbrace{q_0}_{\text{state}} \underbrace{\omega_1 \omega_2 \cdots \omega_n}_{\text{tape contents}}.$$

Generically, a configuration has the form uqv, where u is the tape contents left of the head, q is the state, and v is the remainder of the tape, starting with the current square.

A TM accepts a string ω if there exists a sequence of configurations $C_1, C_2, \cdots C_k$ such that

(1) $C_1 = q_0 \omega$, (the start configuration),

(2) C_i yields C_{i+1} for $i < k$,

where C_i yields C_{i+1} if

$$C_i = u_1 \cdots u_m q_i v_1 \cdots v_n, \quad \delta(q_i, v_1) = (q_j, x, L)$$
$$C_{i+1} = u_1 \cdots u_{m-1} q_j u_m x v_2 \cdots v_n$$

(and likewise for $\delta(q_i, v_1) = (q_j, x, R)$); *

(3) C_k is an accept configuration ($u \, q_{accept} \, v$).

* If our tape head moves L on leftmost square, we stay put.

Example: TM for $J = \{ 0^{2^n} \mid n \geq 0 \}$.

We'll specify a TM M_1 that recognizes J.

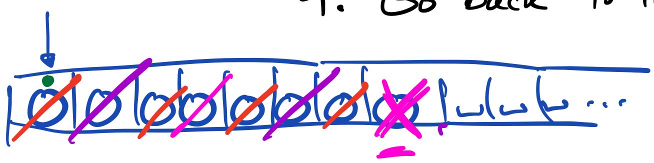
M_1 = "On input ω:

1. Read input left to right, crossing off every other 0, (starting with first char)

$(\Sigma = \{0\},\ \Gamma = \{0, \cancel{0}\})$.

$\Gamma \supseteq \{0, \cancel{0}, \dot{0}, \dot{\cancel{0}}\}$

2. If we saw off exactly one uncrossed zero, <u>accept</u>.
3. If we saw an odd number of zeros, <u>reject</u>.
4. Go back to the leftmost square and repeat from (1)."

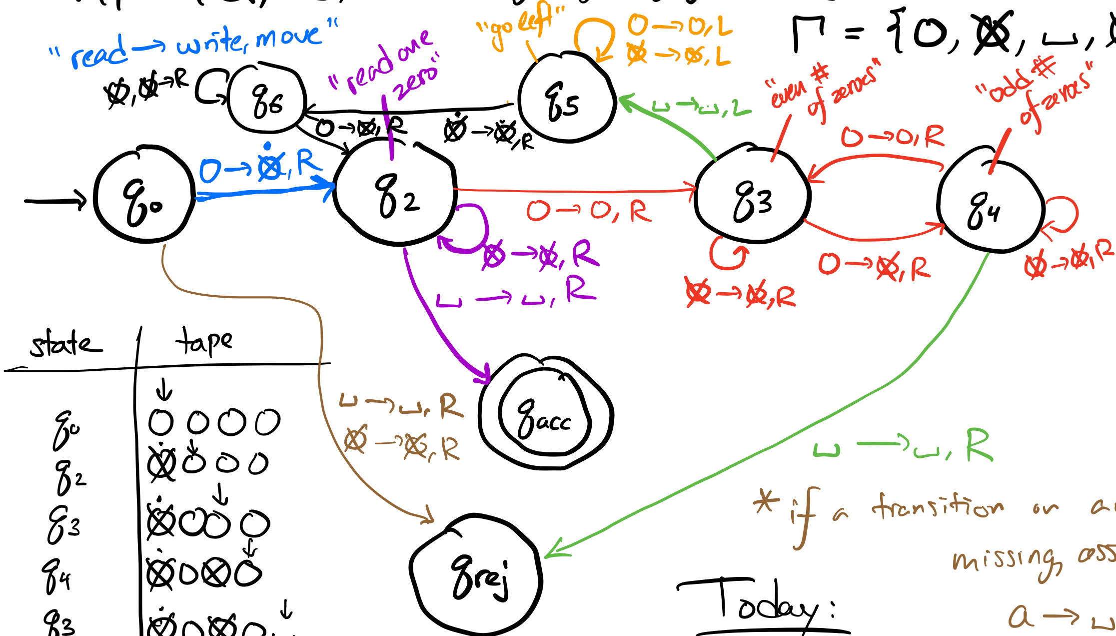


Back at 3:56

State diagram for $J = \{0^{2^n} \mid n \geq 0\}$.

$M_1 = \{Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}\}$. $\Sigma = \{0\}$

$\Gamma = \{0, \cancel{0}, \sqcup, \dot{\cancel{0}}\}$



state	tape
q_0	0 0 0 0
q_2	$\dot{\cancel{0}}$ 0 0 0
q_3	$\dot{\cancel{0}}$ 0 $\cancel{0}$ 0
q_4	$\dot{\cancel{0}}$ 0 $\cancel{0}$ 0
q_3	$\dot{\cancel{0}}$ 0 $\cancel{0}$ 0 $\sqcup$
$q_5 \times 3$	$\dot{\cancel{0}}$ 0 $\cancel{0}$ 0

* if a transition on $a \in \Sigma$ is missing, assume $a \to \sqcup, R$, to reject.

Today:

1. CFPL
2. TMs

Notes:

HW3 due today

HW4 due 6/20 (Tues)

Quick check-in survey on the website.